

Axial anomaly and index of the overlap hypercube operator

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Abstract. The overlap hypercube fermion is constructed by inserting a lattice fermion with hypercubic couplings into the overlap formula. One obtains an exact Ginsparg–Wilson fermion, which is more complicated than the standard overlap fermion, but which has improved practical properties and is of current interest for use in numerical simulations. Here we deal with conceptual aspects of the overlap hypercube Dirac operator. Specifically, we evaluate the axial anomaly and the index, demonstrating that the correct classical continuum limit is recovered. Our derivation is non-perturbative and therefore valid in all topological sectors. At the non-perturbative level this result had previously only been shown for the standard overlap Dirac operator with Wilson kernel. The new techniques which we develop to accomplish this also for hypercubic kernels are of a general nature and have the potential to be extended to overlap Dirac operators with even more general kernels.

1 Introduction

In gauge theories with fermions, the index of a Dirac operator plays an important rôle. It is given as the difference of the number of left- and right-handed zero modes, and due to the Atiyah–Singer index theorem [1] it can be identified with the topological charge of the gauge field. Therefore the same quantity also provides the integrated axial anomaly.

On the lattice it is a non-trivial question if these quantities can be recovered. In particular it is not clear a priori if we obtain the correct axial anomaly in the classical continuum limit. The traditional lattice formulation by means of the Wilson–Dirac operator D_W does not allow for the index theorem to be adapted: lattice gauge configurations do not have natural topological sectors, and the Wilson fermion does not have exact zero modes. Nevertheless the axial anomaly can be reproduced [2]; a necessary condition for this property is the absence of species doublers.

The situation is different for overlap fermions [3]. They have good chiral properties according to their origin from the overlap formalism [4, 5], which are reflected by the fact that their lattice Dirac operator D_{ov} obeys the Ginsparg–Wilson relation (GWR) [6, 7]. In an even dimension $2n$ it reads¹

$$D_{ov}\gamma_{2n+1} + \gamma_{2n+1}D_{ov} = \frac{a}{m}D_{ov}\gamma_{2n+1}D_{ov} . \quad (1.1)$$

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¹ We refer to a Euclidean lattice of spacing a , and the fermions belong to some unitary representation of the (unspecified) gauge group.

Here m is a parameter which controls topological properties and the number of fermion species described by the corresponding lattice fermion action. The GWR turns into the standard condition for chiral symmetry in the continuum limit.

Moreover, even on the lattice an exact chiral symmetry exists [8]. It is lattice modified by a local term of $O(a)$. Explicitly, the variation of the spinor fields under the lattice modified chiral transformation can be written in the form [8]

$$\begin{aligned} \delta\bar{\psi} &= \bar{\psi} \left(1 - \frac{a}{2m} D_{ov} \right) \gamma_{2n+1} , \\ \delta\psi &= \gamma_{2n+1} \left(1 - \frac{a}{2m} D_{ov} \right) \psi , \end{aligned} \quad (1.2)$$

which leaves the lattice action invariant.

The Ginsparg–Wilson relation excludes additive mass renormalization. The index is well defined, since the exact zero modes have a definite chirality [7].² The question arises if the correct axial anomaly is reproduced in the continuum limit. In fact, the existence of some anomaly is obvious since the fermionic measure is not invariant under the transformation (1.2). The resulting anomaly takes the form

$$\mathcal{A}(x) = 2iq(x) , \quad (1.3)$$

² In the case of the overlap Dirac operator the index coincides with the overlap topological charge [4]. For the fixed point fermion, which also solves the GWR, it coincides with the classically perfect topological charge [7].

where $q(x)$ is the topological charge density,

$$q(x) = -\frac{1}{2m} \text{tr} \left(\gamma_{2n+1} D_{\text{ov}} \right) (x, x) . \quad (1.4)$$

Generally we define the density $\mathcal{O}(x, y)$ of a lattice operator \mathcal{O} by

$$\mathcal{O}\psi(x) = a^{2n} \sum_y \mathcal{O}(x, y) \psi(y) . \quad (1.5)$$

Then the operator trace can be expressed as

$$\text{Tr} \mathcal{O} = a^{2n} \sum_x \text{tr} \mathcal{O}(x, x) , \quad (1.6)$$

where tr sums over spinor and gauge indices. In particular the index is given by

$$\text{index}(D_{\text{ov}}) = -\frac{1}{2m} \text{Tr} (\gamma_{2n+1} D_{\text{ov}}) = a^{2n} \sum_x q(x) . \quad (1.7)$$

The question if one obtains the correct expression for the axial anomaly has been studied in a number of papers. This issue was first addressed in [6], where a perturbative evaluation of the anomaly was given for a general Dirac operator satisfying the Ginsparg–Wilson relation.³ Furthermore, a perturbative evaluation of the axial anomaly for very general lattice Dirac operators has been carried out in [10]. In all cases the anomaly was found to reduce to the correct continuum expression if the operator is local and free of species doubling. However, the use of perturbation theory restricts the validity of these arguments to the sector of topologically trivial gauge fields. In fact, the question for which Ginsparg–Wilson fermions the correct anomaly holds also in topological sectors different from zero seems to be non-trivial in the light of [11], which presents an example where this is apparently not the case. So far, the *only* Ginsparg–Wilson operator for which the correct continuum limits of the axial anomaly and index have been established *non-perturbatively* is the standard overlap Dirac operator, which is given by substituting the Wilson–Dirac operator $D = D_{\text{W}}$ into the overlap formula

$$D_{\text{ov}} = \frac{m}{a} \left[1 + A/\sqrt{A^\dagger A} \right] , \quad A = D - \frac{m}{a} . \quad (1.8)$$

The continuum limit of the axial anomaly in this case has been studied explicitly in [12–15]. In particular, a rigorous non-perturbative demonstration that the anomaly and the index have the correct continuum limit in all topological sectors when m is in the physical (doubler-free) region was given in [13, 14].

Currently there is interest in non-standard overlap Dirac operators obtained by inserting more general lattice Dirac operators D into the overlap formula (1.8). The background and motivation for this is discussed further below. In particular, overlap Dirac operators where the

input D is a *hypercubic fermion* (HF) operator have been the focus of attention.⁴ In the light of this development it is pertinent to show at the non-perturbative level that the axial anomaly and index for these non-standard overlap Dirac operators also have the correct continuum limit in all topological sectors. As mentioned above, this has so far only been shown for the standard overlap operator where the input D is the Wilson–Dirac operator. The purpose of the present paper is to establish this result at the non-perturbative level for non-standard overlap Dirac operators of specific current interest, namely those for which the input D in the overlap formula is a HF operator. To do this we follow the rigorous non-perturbative approach of [13, 14] for the standard overlap Dirac case, and take inspiration from [15], where a topological description of the anomaly coefficient as the degree of a certain map was derived which greatly facilitates its evaluation. However, the key technical parts of the arguments in those papers are specific to the standard overlap case – they rely on the explicit form of the Wilson kernel and do not have a straightforward generalization to more general kernels. Therefore we have had to develop new techniques and formulae to handle the more general HF case. In fact, our techniques are of a general nature and have the potential to be used for even more general kernels. (The case of overlap Dirac operator with completely general kernel has further complications though, and is postponed to a later analysis.)

Let us now discuss the background and motivation for considering the non-standard overlap operators mentioned above. Their use was suggested in [16]. The motivation is to improve other properties of the overlap Dirac operator – beyond chirality – which are also of importance for a lattice fermion formulation, such as the quality of scaling, locality and the approximate rotation invariance. We emphasize that chiral properties of the overlap operator D_{ov} continue to hold for *any* input lattice Dirac operator D (free of species doubling) in (1.8), since D_{ov} satisfies the GWR for any such choice. The basic idea is to construct a short range, doubler-free lattice Dirac operator D for the input into the overlap formula which has the following properties:

- (i) good scaling and approximate rotational invariance, and
- (ii) good chirality in the sense that D is an approximate solution of the GWR.

The property (ii) suggests that the overlap Dirac operator D_{ov} obtained from inserting D into the overlap formula will inherit to a large degree the properties (i) of D , and will furthermore have good locality properties. To see this, note that if D is an exact solution of the GWR then the overlap formula just gives D back again: $D_{\text{ov}} = D$ [16]. It is known that sensible ultra-local lattice Dirac operators cannot exactly satisfy the GWR [17]; but approximate solutions are possible, and for these we have $D_{\text{ov}} \approx D$, in-

³ This perturbative evaluation was reconsidered in the context of modern developments of the Ginsparg–Wilson relation [9].

⁴ HF operators are generalizations of the Wilson–Dirac operator which couple all sites within a lattice unit hypercube. (Recall that the Wilson–Dirac operator couples only nearest neighbor sites.)

dicating that D_{ov} approximately inherits the properties of D and is also likely to have good locality properties since D is ultra-local.

A specific construction of a short range lattice Dirac D with the properties (i) and (ii) above arises from the perfect action formalism. This formalism produces, in principle, a lattice fermion action free of artifacts in the scaling behavior via the iteration of renormalization group transformations. Moreover, the corresponding lattice Dirac operator satisfies the GWR as well [6]. The construction of the perfect action can only be carried out explicitly for free and for perturbatively interacting fermions [21].⁵ Still, at the non-perturbative level one can construct *approximations* of the *classically* perfect action (or fixed point action) of asymptotically free theories. The scaling artifacts of the fixed point action tend to be very small – as a study in the 2d $O(3)$ model revealed [23] – and the fixed point Dirac operator solves the GWR too [7]. Hence approximations, or truncations, of the fixed point Dirac operator are natural candidates for operators with the properties (i) and (ii) above.⁶ In particular, a *hypercubic* approximation of the fixed point Dirac operator has been considered, and it was found in numerical studies that the use of this operator as input in the overlap formula can significantly improve the scaling, locality and convergence properties of the overlap Dirac operator [18–20]. These properties have been demonstrated in the Schwinger model [18], and also in QCD they have a potential to compensate the additional complication in D_{HF} compared to D_{W} : for the HF described in [19] the locality of D_{ov} is improved by a factor of 2 in the exponential decay compared to the standard overlap fermion.⁷ Also the convergence rate increases significantly. However, to establish the overlap HF operator as a viable Dirac operator for lattice QCD one also has to check the conceptual basis, in particular whether the correct axial anomaly and index are reproduced, and this is the issue that we address in the present paper. Experiments with simpler, non-standard operators inserted in the overlap formula have been performed in [27]. All those formulations are also covered as special cases by the considerations in this paper.

The organization of this paper is as follows. In Sect. 2 we discuss the properties of D_{HF} which are needed to compute its axial anomaly and index in Sect. 3. The conclusions and an outlook on further generalizations are given in Sect. 4.

⁵ In that case, the axial anomaly takes the correct form even at finite lattice spacing [22].

⁶ For direct QCD applications of a truncated fixed point Dirac operator, see [20, 24]. However, the truncation distorts the chiral symmetry; indeed, even for truncated perfect actions the additive mass renormalization can be considerable [25]. Chiral symmetry can be re-imposed though by inserting the truncated fixed point operator into the overlap formula, and this is another motivation to study the overlap operator with truncated fixed point operator as kernel.

⁷ This number refers to quenched QCD at $\beta = 6$, and the corresponding test for the standard overlap fermion was performed in [26].

2 The structure of the hypercube Dirac operator

We are going to use the following conventions for the γ matrices: $(\gamma_\mu)^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_{2n+1} = i^n \gamma_1 \cdots \gamma_{2n}$, so that $(\gamma_{2n+1})^\dagger = \gamma_{2n+1}$ and $\text{tr}(\gamma_{2n+1} \gamma_1 \cdots \gamma_{2n}) = (-2i)^n$. Throughout this paper summation over repeated indices is implied.

What we consider here is the minimally gauged HF–Dirac operator, which we are going to describe now. For techniques to simulate such HF’s in QCD, see [28].

Assume the two lattice sites x and y to belong to the same lattice hypercube, i.e. $|x_\mu - y_\mu| \leq a$ for $\mu = 1, \dots, 2n$. Then we denote by $P(x, y)$ the set of lattice paths of minimal length connecting x and y . All these paths are inside the same hypercube again, their length is $n_{xy} \in \{0, 1, \dots, 2n\}$, and the number of such paths is $n_{xy}!$. Let σ be one of these paths. In the presence of a compact lattice gauge field $U_\mu(x)$ we denote by $U(\sigma)$ the product of link variables along the path σ . We also define the sign function $\epsilon(t) = \text{sign } t$ for $t \neq 0$, and $\epsilon(0) = 0$. Then the minimally gauged HF operator can be written as

$$D_{\text{HF}} = \frac{1}{a} \left(\gamma_\mu \rho_\mu + \lambda \right), \quad (2.1)$$

$$(\rho_\mu)_{xy} = \kappa_{n_{xy}} \frac{\epsilon(x_\mu - y_\mu)}{n_{xy}!} \sum_{\sigma \in P(x, y)} U(\sigma),$$

$$\lambda_{xy} = \lambda_{n_{xy}} \frac{1}{n_{xy}!} \sum_{\sigma \in P(x, y)} U(\sigma). \quad (2.2)$$

The parameters $\kappa_1, \dots, \kappa_{2n}$ and $\lambda_0, \lambda_1, \dots, \lambda_{2n}$ are coupling constants. Here we stick to the simple γ -structure of the Wilson fermion.⁸ The vector term ρ_μ alone characterizes a generalized naive fermion, which would by itself generate species doublers. The scalar term λ can be considered a generalized Wilson term which removes these doublers (respectively attaches a mass of the cutoff scale to them) for suitable couplings; see below. $\kappa_{n_{xy}}$ and $\lambda_{n_{xy}}$ couple one site x to $2^{n_{xy}}$ sites y .⁹

D_{HF} has the correct formal continuum limit with vanishing bare mass precisely when

$$\sum_{j=1}^{2n} 2^j \begin{bmatrix} 2n-1 \\ j-1 \end{bmatrix} \kappa_j = 1, \quad \sum_{j=0}^{2n} 2^j \begin{bmatrix} 2n \\ j \end{bmatrix} \lambda_j = 0. \quad (2.3)$$

At a finite β one deviates from these constraints and amplifies each coupling (except for λ_0) in order to compensate its suppression by the link variable [19]. However, in the current context we do impose the above constraints because they have to be restored in the classical continuum limit. They can be used to eliminate κ_1 and λ_0 ; then D_{HF} contains $4n - 1$ free parameters $\kappa_2, \dots, \kappa_{2n}$; $\lambda_1, \dots, \lambda_{2n}$.

⁸ HF’s with a more general Clifford algebra have also been used in QCD simulation; see [25, 29, 20].

⁹ The usual Wilson–Dirac operator with Wilson parameter r and bare mass m_0 is recovered by setting $\kappa_1 = 1/2$, $\lambda_0 = 2nr + am_0$, $\lambda_1 = -r/2$ and $\kappa_j = \lambda_j = 0$ for $j \geq 2$.

For our purposes it is useful to express D_{HF} in a coordinate-free way as follows. We use the parallel transporters

$$\begin{aligned} T_{+\mu}(x, y) &= U_\mu(x) \delta_{x, y - a\hat{\mu}}, \\ T_{-\mu}(x, y) &= U_\mu^\dagger(x - \hat{\mu}) \delta_{x - a\hat{\mu}, y}, \end{aligned} \quad (2.4)$$

(where $\hat{\mu}$ is the unit vector in the μ direction) to define the Hermitian operators

$$S_\mu = \frac{1}{2i}(T_{+\mu} - T_{-\mu}), \quad C_\mu = \frac{1}{2}(T_{+\mu} + T_{-\mu}). \quad (2.5)$$

Moreover we introduce the following notation for the symmetrized product of operators,

$$[\mathcal{O}_1 \dots \mathcal{O}_p]_{\text{sym}} = \frac{1}{p!} \sum_{\alpha} \mathcal{O}_{\alpha(1)} \dots \mathcal{O}_{\alpha(p)}, \quad (2.6)$$

where the sum runs over all permutations α of $\{1, \dots, p\}$. Then (2.2) can be re-expressed as

$$\begin{aligned} -i \rho_\mu &= S_\mu \\ &- \sum_{p=2}^{2n} 2^p \kappa_p \sum_{\nu_2 < \dots < \nu_p; \nu_j \neq \mu \forall j} [S_\mu (1 - C_{\nu_2} \dots C_{\nu_p})]_{\text{sym}}, \\ \lambda &= \sum_{p=1}^{2n} 2^p (-\lambda_p) \sum_{\nu_1 < \dots < \nu_p} [1 - C_{\nu_1} \dots C_{\nu_p}]_{\text{sym}}. \end{aligned} \quad (2.7)$$

Note that $\rho_\mu^\dagger = -\rho_\mu$, $\lambda^\dagger = \lambda$; hence D_{HF} satisfies γ_{2n+1} -Hermiticity,

$$D_{\text{HF}}^\dagger = \gamma_{2n+1} D_{\text{HF}} \gamma_{2n+1}. \quad (2.8)$$

We now consider the zero modes of the free field ($U = 1$) “naive” HF-Dirac operator $\frac{1}{a} \gamma_\mu \rho_\mu$ and their “masses” provided by the scalar term $\frac{1}{a} \lambda$ in (2.1). The free field momentum representation of S_μ , C_μ (i.e. their eigenvalues for the plane wave eigenfunction $e^{ikx/a}$) is obvious from (2.5):

$$S_\mu(k) = \sin(k_\mu) \equiv s_\mu, \quad C_\mu(k) = \cos(k_\mu) \equiv c_\mu. \quad (2.9)$$

Hence the free field momentum representations¹⁰ of ρ_μ and λ are

$$\begin{aligned} \rho_\mu(k) & \\ &= i s_\mu \left[1 - \sum_{p=2}^{2n} 2^p \kappa_p \sum_{\nu_2 < \dots < \nu_p; \nu_j \neq \mu \forall j} (1 - c_{\nu_2} \dots c_{\nu_p}) \right], \end{aligned} \quad (2.10)$$

$$\lambda(k) = \sum_{p=1}^{2n} 2^p (-\lambda_p) \sum_{\nu_1 < \dots < \nu_p} (1 - c_{\nu_1} \dots c_{\nu_p}). \quad (2.11)$$

The former vanishes when $s_\mu = 0$, so the naive HF-Dirac operator has the usual zero mode at $k = 0$ and the familiar $2^{2n} - 1$ “doubler” zero modes for k at the corners of

¹⁰ Note that k represents a *re-scaled* momentum, which is 2π periodic at any lattice spacing a . Nevertheless we denote k simply as “the momentum”.

the Brillouin zone, just as in the case of the usual naive Dirac operator. However, in addition to these, there can be other zero modes corresponding to vanishing of the factor in the square brackets on the right-hand side of (2.10). It can vanish for some momenta k with components different from 0 and π , when $\kappa_2, \dots, \kappa_{2n}$ are in certain regions of the parameter space. These zero modes correspond to new “exotic” spurious fermion species: if such a zero mode occurs at $k = k^{(0)}$ we set $k = k^{(0)} + k'$ and find that the leading order term in the expansion of $\gamma_\mu \rho_\mu(k)$ around $k^{(0)}$ is $\sim \gamma_\mu \sum_{\nu \neq \mu} k'_\nu$. The corresponding propagator does not describe a usual Dirac fermion species. These exotic species are excluded though, if the parameters $\kappa_2, \dots, \kappa_{2n}$ satisfy

$$\chi_\mu[\kappa_2, \dots, \kappa_{2n}](k) \quad (2.12)$$

$$\equiv \sum_{p=2}^{2n} 2^p \kappa_p \sum_{\nu_2 < \dots < \nu_p; \nu_j \neq \mu \forall j} (1 - c_{\nu_2} \dots c_{\nu_p}) < 1 \quad \forall k.$$

(Note that if this is satisfied for a particular index μ then it is satisfied for all $\mu = 1, \dots, 2n$. Also, since $\chi_\mu = 0$ at $k = 0$, the condition $\chi_\mu < 1$ is the same as $\chi_\mu \neq 1$.)

At this point it is natural to ask: what are the values of $\kappa_1, \dots, \kappa_{2n}$ that are of interest in practice, and do they satisfy (2.12)? The values of $\lambda_1, \dots, \lambda_{2n}$ used in practice are also relevant here since they determine $\lambda(k)$ and hence the masses of the doubler fermion species. As discussed in the introduction, one of the main aims in choosing the coupling parameters is to make D_{HF} as close as possible to satisfying the GW relation. The procedure of truncating perfect fermions (described in Sect. 1) led to the following values for the couplings [25, 16] in dimensions $2n = 2$ and $2n = 4$:¹¹

$$\kappa_1 = 0.309, \quad \kappa_2 = 0.095 \quad (2.13)$$

$$\lambda_0 = 1.490, \quad \lambda_1 = -0.245, \quad \lambda_2 = -0.128;$$

$2n = 4$:

$$\begin{aligned} \kappa_1 &= 0.137, \quad \kappa_2 = 0.032, \quad \kappa_3 = 0.011, \quad \kappa_4 = 0.005, \\ \lambda_0 &= 1.853, \quad \lambda_1 = -0.061, \quad \lambda_2 = -0.030, \\ \lambda_3 &= -0.016, \quad \lambda_4 = -0.008. \end{aligned} \quad (2.14)$$

In two dimensions the left-hand side of (2.12) is

$$\chi_\mu^{(2n=2)} = 4\kappa_2(1 - c_\nu) \epsilon_{\mu\nu}. \quad (2.15)$$

The maximum of this, attained at $c_\nu = -1$, is $8\kappa_2 = 0.76$ for the coupling values in (2.13); hence (2.12) is satisfied. In the dimension four case the left-hand side of (2.12) can be re-written as

¹¹ We give the values to 3 decimal places; they are given to higher precision in [25, 16].

Table 1. Masses in dimensions 2 and 4, with couplings given by (2.13)–(2.14), in lattice units, and the masses of the corresponding modes of the Wilson–Dirac operator with Wilson parameter r

$2n = 2:$					
N_π	0	1	2		
M_{HF}	0	2.004	1.960		
M_{W}	0	$2r$	$4r$		
$2n = 4:$					
N_π	0	1	2	3	4
M_{HF}	0	1.988	1.960	1.964	2.000
M_{W}	0	$2r$	$4r$	$6r$	$8r$

$$\begin{aligned} \chi_\mu^{(2n=4)} &= 12\kappa_2 + 24\kappa_3 + 16\kappa_4 \\ &+ 16 \frac{\kappa_3^2}{\kappa_2} \left(\alpha^3 - (\alpha + c_{\nu_2})(\alpha + c_{\nu_3})(\alpha + c_{\nu_4}) \right) \\ &- 16 \left(\kappa_4 - \frac{\kappa_3^2}{\kappa_2} \right) c_{\nu_2} c_{\nu_3} c_{\nu_4}, \end{aligned} \tag{2.16}$$

where $\{\mu, \nu_2, \nu_3, \nu_4\} = \{1, 2, 3, 4\}$ and $\alpha = \kappa_2/(2\kappa_3)$. From this we see that when $\kappa_2 \geq 2\kappa_3$ (i.e. $\alpha \geq 1$) and $\kappa_4 \geq \kappa_3^2/\kappa_2$ the maximum of $\chi_\mu^{(2n=4)}$ is attained at $c_{\nu_2} = c_{\nu_3} = c_{\nu_4} = -1$. For the coupling values (2.14) this maximum is 0.93, hence (2.12) is again satisfied.

Let $\{k^{(j)}\}$ denote the momenta of the zero modes for the free field naive HF–Dirac operator. The mass of such a mode, provided by the scalar term in D_{HF} , is $M^{(j)}/a$ where $M^{(j)} \equiv \lambda(k^{(j)})$. To avoid species doubling in the full HF–Dirac operator we impose the requirement on $\lambda_1, \dots, \lambda_{2n}$ that $\lambda(k^{(j)}) > 0$ for $k^{(j)} \neq 0$. From (2.11) we see that a sufficient condition for this is $\lambda_1 < 0$ and $\lambda_p \leq 0 \ \forall p=2, \dots, 2n$, which holds for the coupling values in (2.13)–(2.14) (and of course also for the Wilson–Dirac operator). For the usual zero and doubler modes, characterized by $s_\mu = 0 \ \forall \mu$, i.e. $k_\mu = 0$ or π for each μ , let N_π denote the number of k_μ equal to π . Then, from (2.11), the mass M/a of the mode is seen to depend only on N_π as follows:

$$M(N_\pi) = \sum_{p=1}^{2n} 2^{p+1} (-\lambda_p) \sum_{q=1}^p \binom{2n-N_\pi}{p-q} \binom{N_\pi}{q}, \tag{2.17}$$

with $\binom{N_\pi}{q} \equiv 0$ for $q > N_\pi$. From this the masses in dimensions 2 and 4, with couplings given by (2.13)–(2.14), can be determined (see also Figs. 1 and 2 in [16]). We list them in Table 1 in lattice units. For comparison we also list the masses of the corresponding modes of the Wilson–Dirac operator with Wilson parameter r .

The HF doubler masses are all close to 2 in lattice units, reflecting the fact that the free field D_{HF} with the coupling values from (2.13) respectively (2.14) are good approximate solutions to the GW relation (since for exact GW solutions the eigenvalues lie on the circle in the complex plane centered at $(1/a, 0)$ with radius $1/a$). We

also remark that, unlike in the Wilson case, M_{HF} does not always increase with increasing N_π .

3 The continuum limit of the axial anomaly for the overlap-HF–Dirac operator

For a given value of the parameter m , the momenta of the zero modes of the free field D_{ov} are the $k^{(j)}$ with $M^{(j)} < m$ (both defined in Sect. 2). Hence the parameter region in which D_{ov} has a physical zero mode and no doublers is $0 < m < \min\{M^{(j)} \neq 0\}$.

Our aim now is to evaluate the classical continuum limit of the topological charge density $q(x)$ given by (1.4), or equivalently, the axial anomaly $\mathcal{A}(x) = 2iq(x)$, of the overlap-HF–Dirac operator. Specifically, we consider the situation where D_{ov} is coupled to the lattice transcript of a smooth continuum gauge field $A = A_\mu(x)dx^\mu$, i.e. the link variables are

$$U_\mu(x) = T \exp \left(a \int_0^1 A_\mu(x + (1-t)a\hat{\mu}) dt \right), \tag{3.1}$$

where T implies t -ordering. We will derive the following result:

If $m \neq M^{(j)} \ \forall j$, then

$$\lim_{a \rightarrow 0} q(x) = I(\kappa_2, \dots, \kappa_{2n}; \lambda_1, \dots, \lambda_{2n}; m) q_{\text{cont}}(x), \tag{3.2}$$

where

$$\begin{aligned} q_{\text{cont}}(x) &= \frac{1}{(2\pi i)^n n!} \\ &\cdot \frac{1}{2^n} \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left[F_{\mu_1 \mu_2}(x) \cdots F_{\mu_{2n-1} \mu_{2n}}(x) \right] \end{aligned} \tag{3.3}$$

is the continuum topological charge density and $I(\kappa_2, \dots, \lambda_{2n}; m)$ is the degree of a certain map $\Theta : T^{2n} \rightarrow S^{2n}$, given in (3.22) below. In particular, $I(\kappa_2, \dots, \lambda_{2n}; m) = 1$ holds for m in the physical (doubler-free) region $0 < m < \min\{M^{(j)} \neq 0\}$. Thus, for m in this region, $q(x)$ and the axial anomaly reduce to the correct continuum expressions in the classical continuum limit. Furthermore, when the parameters $\kappa_2, \dots, \kappa_{2n}$ satisfy the constraint (2.12), then

$$\begin{aligned} I(\kappa_2, \dots, \kappa_{2n}; \lambda_1, \dots, \lambda_{2n}; m) &= \sum_{\{N_\pi : M(N_\pi) < m\}} \binom{2n}{N_\pi} (-1)^{N_\pi}. \end{aligned} \tag{3.4}$$

We first derive the result in the infinite volume, i.e. on a hypercubic lattice on \mathbb{R}^{2n} , and thereafter we discuss the finite volume $2n$ -torus case. The expression (1.4) can be re-written as

$$q(x) = -\frac{1}{2} \text{tr} \left(\frac{H_m}{\sqrt{H_m^2}} \right) (x, x), \tag{3.5}$$

where

$$H_m = \gamma_{2n+1}(aD_{\text{HF}} - m) = \gamma_{2n+1}(\gamma_\mu \rho_\mu + \lambda - m) \quad (3.6)$$

is the Hermitian HF–Dirac operator (normalized by $1/a$). We proceed as in the Wilson case treatment of [13,14] by expanding $(H_m^2)^{-1/2}$ as a power series. First, H_m^2 is decomposed as

$$\begin{aligned} H_m^2 &= L - V, \\ L &= -\rho^2 + (\lambda - m)^2, \\ V &= \gamma_\mu[\rho_\mu, \lambda] + \frac{1}{2}\gamma_\mu\gamma_\nu[\rho_\mu, \rho_\nu]. \end{aligned} \quad (3.7)$$

As in the Wilson case we observe $\|V\| \sim O(a^2)$, which is a consequence of the property $\|[T_{\pm\mu}, T_{\pm\nu}]\| \sim O(a^2)$ and (2.7). Furthermore, a lower bound $0 < b < L$ exists when the lattice is sufficiently fine.¹² This implies that $\|L^{-1}V\| \sim O(a^2)$ and consequently $(H_m^2)^{-1/2} = (L[1 - L^{-1}V])^{-1/2}$ can be expanded as a power series in $L^{-1}V$ when the lattice spacing a is sufficiently small. This was done in the Wilson case [13,14] using an integral representation of the inverse square root. (Note that the integral representation is needed since L and V do in general not commute.) The argument relies only on general properties of L and V which continue to hold in the present HF case; the treatment in [13,14] generalizes straightforwardly to the HF and to arbitrary even dimension. Substituting the resulting expansion into (3.5) and using the lattice δ -function

$$\delta_x = \int_{-\pi}^{\pi} \frac{d^{2n}k}{(2\pi)^{2n}} e^{-ikx/a} \phi_k, \quad \phi_k(y) \equiv e^{iky/a}, \quad (3.8)$$

to express $q(x)$ as an integral over momentum space, one obtains¹³

$$\begin{aligned} q(x) &= -\frac{c(n)}{2a^{2n}} \int_{-\pi}^{\pi} \frac{d^{2n}k}{(2\pi)^{2n}} \frac{\text{tr}(H_m(k) e^{-ikx/a} V^n e^{ikx/a})}{L(k)^{n+1/2}} \\ &\quad + O(a), \end{aligned} \quad (3.9)$$

where $H_m(k)$ and $L(k)$ are the free field momentum representations of H_m and L , and

$$c(n) = \frac{1}{n!} \left. \frac{d^n}{dt^n} (1-t)^{-1/2} \right|_{t=0} = \frac{(2n)!}{2^{2n}(n!)^2}. \quad (3.10)$$

To evaluate the limit $a \rightarrow 0$ of (3.9) we start from the following general observations,

¹² This was established in the Wilson case for restricted values of m in [26,30], and later for general m in [31]. The result will be generalized to the present HF case, and more general cases, in [32].

¹³ To derive (3.9) we have used the fact that the trace of the product of γ_{2n+1} with the product of less than $2n$ γ matrices vanishes. The factor $1/a^{2n}$ in (3.9) originates from the a^{2n} in the operator representation $\mathcal{O}\psi(x) = a^{2n} \sum_y \mathcal{O}(x,y)\psi(y)$ ($\mathcal{O} = H_m/\sqrt{H_m^2}$). Hence the first term is of $O(1)$.

$$\begin{aligned} e^{-ikx/a} [T_{\pm\mu}, T_{\pm\nu}] e^{ikx/a} &= a^2 F_{\mu\nu}(x) e^{i(\pm k_\mu \pm k_\nu)} + O(a^3), \\ e^{-ikx/a} [T_{\pm\mu}, T_{\mp\nu}] e^{ikx/a} &= -a^2 F_{\mu\nu}(x) e^{i(\pm k_\mu \mp k_\nu)} + O(a^3), \end{aligned} \quad (3.11)$$

which imply

$$\begin{aligned} e^{-ikx/a} [S_\mu, S_\nu] e^{ikx/a} &= -a^2 F_{\mu\nu}(x) c_\mu c_\nu + O(a^3), \\ e^{-ikx/a} [S_\mu, C_\nu] e^{ikx/a} &= a^2 F_{\mu\nu}(x) c_\mu s_\nu + O(a^3), \\ e^{-ikx/a} [C_\mu, C_\nu] e^{ikx/a} &= -a^2 F_{\mu\nu}(x) s_\mu s_\nu + O(a^3), \end{aligned} \quad (3.12)$$

with the terms defined in (2.9). In the following we denote the free field momentum representation of a general lattice operator X by $X(k)$. Then the relations (3.12) can be expressed collectively as

$$\begin{aligned} e^{-ikx/a} [X, Y] e^{ikx/a} &= -a^2 F_{\alpha\beta}(x) \partial_\alpha X(k) \partial_\beta Y(k) + O(a^3) \end{aligned} \quad (3.13)$$

for $X = S_\mu, C_\mu$ and $Y = S_\nu, C_\nu$. In fact, this relation continues to hold when X and Y are general polynomials of the S_μ and the C_μ . Since ρ_μ and λ are such polynomials we can apply (3.13) to $e^{-ikx/a} V^n e^{ikx/a}$ in the expression (3.9) to obtain

$$\begin{aligned} \text{tr}(H_m(k) e^{-ikx/a} V^n e^{ikx/a}) &= i^n a^{2n} \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\alpha_1 \alpha_2}(x) \cdots F_{\alpha_{2n-1} \alpha_{2n}}(x) \right) \\ &\quad \times \left[(\lambda(k) - m) \partial_{\alpha_1} \rho_{\mu_1}(k) \cdots \partial_{\alpha_{2n}} \rho_{\mu_{2n}}(k) \right. \\ &\quad \left. - 2n \partial_{\alpha_1} \rho_{\mu_1}(k) \cdots \partial_{\alpha_{2n-1}} \rho_{\mu_{2n-1}}(k) \partial_{\alpha_{2n}} \lambda(k) \rho_{\mu_{2n}}(k) \right] \\ &\quad + O(a^{2n+1}). \end{aligned} \quad (3.14)$$

We now note the two identities which will be crucial for our further considerations:

$$\begin{aligned} \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\alpha_1 \alpha_2}(x) \cdots F_{\alpha_{2n-1} \alpha_{2n}}(x) \right) &\times \partial_{\alpha_1} \rho_{\mu_1}(k) \cdots \partial_{\alpha_{2n}} \rho_{\mu_{2n}}(k) \\ &= \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\mu_1 \mu_2}(x) \cdots F_{\mu_{2n-1} \mu_{2n}}(x) \right) \epsilon_{\alpha_1, \dots, \alpha_{2n}} \\ &\times \partial_{\alpha_1} \rho_1(k) \cdots \partial_{\alpha_{2n}} \rho_{2n}(k), \end{aligned} \quad (3.15)$$

$$\begin{aligned} \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\alpha_1 \alpha_2}(x) \cdots F_{\alpha_{2n-1} \alpha_{2n}}(x) \right) &\times \partial_{\alpha_1} \rho_{\mu_1}(k) \cdots \partial_{\alpha_{2n-1}} \rho_{\mu_{2n-1}}(k) \partial_{\alpha_{2n}} \lambda(k) \rho_{\mu_{2n}}(k) \\ &= \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\mu_1 \mu_2}(x) \cdots F_{\mu_{2n-1} \mu_{2n}}(x) \right) \\ &\times \sum_{p=1}^{2n} (-1)^p \rho_p(k) \epsilon_{\alpha_0 \alpha_1 \dots \alpha_{p-1} \alpha_{p+1} \dots \alpha_{2n}} \\ &\times \partial_{\alpha_0} \lambda(k) \partial_{\alpha_1} \rho_1(k) \cdots \\ &\quad \partial_{\alpha_{p-1}} \rho_{p-1}(k) \partial_{\alpha_{p+1}} \rho_{p+1}(k) \cdots \partial_{\alpha_{2n}} \rho_{2n}(k). \end{aligned} \quad (3.16)$$

These combinatorial identities rely only on the facts that the $\rho_\mu(k)$ and $\lambda(k)$ all commute, on $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$, and on the cyclic property of the trace. Replacing the

left-hand sides of these identities by the right-hand sides in (3.14), and introducing the real-valued functions

$$\begin{aligned} \theta_0(k) &= -[\lambda(k) - m], & \theta_\mu(k) &= -i\rho_\mu(k) \\ & & (\mu &= 1, \dots, 2n), \end{aligned} \quad (3.17)$$

we arrive at

$$\begin{aligned} & \text{tr} \left[H_m(k) e^{-ikx/a} V^n e^{ikx/a} \right] dk_1 \wedge \dots \wedge dk_{2n} \\ &= -(-i)^n a^{2n} \epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} \left(F_{\mu_1 \mu_2}(x) \dots F_{\mu_{2n-1} \mu_{2n}}(x) \right) \\ & \times \sum_{p=0}^{2n} (-1)^p \theta_p d\theta_0 \wedge \dots \wedge d\theta_{p-1} \wedge d\theta_{p+1} \wedge \dots \wedge d\theta_{2n} \\ & + O(a^{2n+1}), \end{aligned} \quad (3.18)$$

where $d\theta_j \equiv \partial_\alpha \theta_j dk_\alpha$ is the exterior derivative of θ_j . Substituting this into (3.9) and re-writing L from (3.7) as

$$L(k) = -\rho^2(k) + [\lambda(k) - m]^2 = \sum_{p=0}^{2n} \theta_p(k)^2 = |\theta(k)|^2, \quad (3.19)$$

we obtain $\lim_{a \rightarrow 0} q(x) = I(\kappa_2, \dots, \lambda_{2n}; m) q_{\text{cont}}(x)$, as we claimed in (3.2), with

$$\begin{aligned} & I(\kappa_2, \dots, \lambda_{2n}; m) \\ &= \frac{1}{2} c(n) \frac{n!}{\pi^n} \int_{[-\pi, \pi]^{2n}} \frac{1}{|\theta|^{2n+1}} \\ & \times \sum_{p=0}^{2n} (-1)^p \theta_p d\theta_0 \wedge \dots \wedge d\theta_{p-1} \wedge d\theta_{p+1} \wedge \dots \wedge d\theta_{2n}. \end{aligned} \quad (3.20)$$

A little calculation shows that the integrand here can be re-written as

$$\begin{aligned} & \sum_{p=0}^{2n} (-1)^p \frac{\theta_p}{|\theta|} d \left(\frac{\theta_0}{|\theta|} \right) \wedge \dots \wedge d \left(\frac{\theta_{p-1}}{|\theta|} \right) \wedge d \left(\frac{\theta_{p+1}}{|\theta|} \right) \wedge \\ & \dots \wedge d \left(\frac{\theta_{2n}}{|\theta|} \right). \end{aligned} \quad (3.21)$$

This is precisely the pull-back to $T^{2n} =]-\pi, \pi]^{2n}$ of the volume form on the unit $2n$ -sphere $S^{2n} \subset \mathbb{R}^{2n+1}$ via the map

$$\Theta : T^{2n} \rightarrow S^{2n}, \quad \Theta(k) := \left(\frac{\theta_0}{|\theta|}, \frac{\theta_1}{|\theta|}, \dots, \frac{\theta_{2n}}{|\theta|} \right). \quad (3.22)$$

Furthermore, the coefficient of the integral in (3.20) turns out to be (recall definition (3.10))

$$\frac{1}{2} c(n) \frac{n!}{\pi^n} = \frac{(2n)!}{2^{2n+1} n! \pi^n} = \frac{1}{\text{Vol}(S^{2n})}, \quad (3.23)$$

where $\text{Vol}(S^{2n})$ is the volume of the unit $2n$ -sphere. Hence expression (3.20) calculates the *degree* of the map (3.22),

$$I(\kappa_2, \dots, \lambda_{2n}; m) = \text{deg}(\Theta). \quad (3.24)$$

This is a generalization of the topological evaluation of the anomaly coefficient given in [15].

If $\frac{\theta}{|\theta|} \in S^{2n} \subset \mathbb{R}^{2n+1}$ is a regular point for the map Θ , then it is a standard topological fact that $\text{deg}(\Theta) = \sum_l s_l$, where $s_l = \pm 1$ is the sign (relative to $dk_1 \wedge \dots \wedge dk_{2n}$) of the integrand of (3.20) evaluated at a pre-image $k^{(l)}$ of $\frac{\theta}{|\theta|}$, and the sum is over all the pre-images (labelled by l). We choose $\frac{\theta}{|\theta|} = (1, 0, \dots, 0)$. Then the pre-images $k^{(l)}$ are precisely the subset of the $k^{(j)}$ introduced in Sect. 2 which satisfy $\theta_0 = -[\lambda(k) - m] > 0$, i.e. for which $M^{(j)} = \lambda(k^{(j)}) < m$. Moreover, the integrand in (3.20) reduces at these momenta to $d\theta_1 \wedge \dots \wedge d\theta_{2n}$. To determine the sign of this term at a given $k^{(j)}$, recall from (2.10) that $\theta_\mu(k) = -i\rho_\mu(k) = S_\mu(k)[1 - \chi_\mu(k)]$ with $\chi_\mu(k)$ as defined in (2.12). It follows that

$$\begin{aligned} & d\theta_1 \wedge \dots \wedge d\theta_{2n} \\ &= \left(\prod_{\mu=1}^{2n} C_\mu(k) \right) \left(\prod_{\nu=1}^{2n} (1 - \chi_\nu(k)) \right) dk_1 \wedge \dots \wedge dk_{2n} \\ & + \text{terms with at least one } S_\mu(k) \text{ factor.} \end{aligned} \quad (3.25)$$

If $0 < m < \min\{M^{(j)} \neq 0\}$, then there is precisely one $k^{(j)}$ for which $M^{(j)} < m$, namely $k^{(j)} = 0$. In this case, since $\chi_\mu(0) = 0 \quad \forall \mu$, (3.25) yields $d\theta_1 \wedge \dots \wedge d\theta_{2n} \Big|_{k=0} = dk_1 \wedge \dots \wedge dk_{2n} \Big|_{k=0}$, i.e. the sign is $+1$, so $\text{deg}(\Theta) = 1$ in this case, as we claimed. Let us now consider the case where the parameters $\kappa_2, \dots, \kappa_{2n}$ satisfy the constraint (2.12), i.e. $\chi_\mu(k) < 1 \quad \forall \mu, k$. Then the $k^{(j)}$ are precisely the k at which $S_\mu(k) = 0 \quad \forall \mu$, so the terms with $S_\mu(k)$ factors in (3.25) vanish and the sign of the remainder is given by $\prod_{\mu=1}^{2n} C_\mu(k^{(j)})$. This sign is $(-1)^{N_\pi^{(j)}}$ where $N_\pi^{(j)}$ is the number of components of $k^{(j)}$ which are equal to π . Recalling from Sect. 2 that $M^{(j)}$ depends only on $N_\pi^{(j)}$ in this case, and noting that the number of $k^{(j)}$ with $N_\pi^{(j)} = N_\pi$ is $\left[\frac{2n}{N_\pi} \right]$, it follows that $\text{deg}(\Theta) = \sum_{\{N_\pi : M(N_\pi) < m\}} \left[\frac{2n}{N_\pi} \right] (-1)^{N_\pi}$. This completes the derivation of the results anticipated in (3.2)–(3.4) in the infinite lattice setting.

The rigorous derivation of the expansion (3.9), carried out along the same lines as in the overlap-Wilson case [13, 14], requires an assumption on the continuum field, namely that $A_\mu(x)$ and its first few partial derivatives are bounded on \mathbb{R}^{2n} . Such bounds are guaranteed to exist if A has a compact support on \mathbb{R}^{2n} . However, having established the result in (3.2)–(3.4) for gauge fields with compact support, it can then be extended to general smooth gauge fields using locality type arguments, in the same way as in the overlap-Wilson case (see (3.45) in [13] and the associated discussion). This relies on the existence of a non-zero lower bound on H_m^2 .

In the finite volume $2n$ -torus setting the momentum integrals in (3.8) and (3.9) become sums, so the derivation given above does not carry over directly to that setting. However, using locality-based arguments one can

show that the finite volume $q(x)$ coincides with the infinite volume $q(x)$ up to exponentially suppressed finite size effects, thereby establishing that the continuum limit results (3.2)–(3.4) continue to hold in the finite volume setting. This was done in the overlap-Wilson case in [14]; the arguments there relied only on general properties and carry over to the present HF case (given the aforementioned lower bound on H_m^2); the details of all this will be given in the general setting in [32].

In the light of the index formula (1.7) it then follows that when the overlap-HF operator is coupled to the lattice transcript of a smooth continuum gauge field A on T^{2n} with topological charge Q , then $\text{index}(D_{\text{ov}})$ reduces to $I(\kappa_2, \dots, \lambda_{2n}; m) Q$ in the limit $a \rightarrow 0$. Thus the HF fermionic topological charge reduces to the continuum topological charge in the classical continuum limit when the parameter m is in the doubler-free region, just as it does in the Wilson case.

4 Summary

We repeat that the previous literature contains the following considerations about the axial anomaly of overlap fermions.

(1) Perturbative considerations show that the correct continuum limit is obtained in the sector of topological charge zero for any overlap operator; see in particular [10].

(2) There was also a rigorous, non-perturbative proof that covers all topological sectors, but it was specific to the case of the simplest standard overlap fermion, which uses the Wilson-Dirac operator as an input [13,14].

The standard overlap operator is widespread in recent simulations. However, there are attempts by various groups to use also non-standard overlap operators [18–20, 27]. The operators used in those works are all included in the class of HF overlap operators. For the latter we have given in this paper a non-perturbative evaluation of the continuum limit of the axial anomaly and index which is valid in all topological sectors.

We have formulated the HF-Dirac operator in $2n$ -dimensional Euclidean space in the form (2.7), which is well-suited to analytic investigations. We used it first to study the dependence of the doubler structure of D_{HF} on its coupling parameters. Then we evaluated the classical continuum limits of the axial anomaly and index of the overlap-HF-Dirac operator, showing that the correct continuum expressions are recovered when parameters are in the physical (doubler-free) region. A noteworthy feature of our continuum limit evaluation is that it relies only on general properties of the HF-Dirac operator and not its explicit form. This is in contrast to the previous evaluations in the Wilson case (a special case of the more general HF structure), which all use the explicit form of the operator. The main new technical observations which our approach is based on are the general relation (3.13) and the identities (3.15)–(3.16). These ingredients allow the continuum form $\epsilon_{\mu_1 \dots \mu_{2n}} \text{tr} F_{\mu_1 \mu_2}(x) \cdots F_{\mu_{2n-1} \mu_{2n}}(x)$ of the axial anomaly to be extracted, and its coefficient to be topologically evaluated as the degree of a map $\Theta : T^{2n} \rightarrow S^{2n}$

using only general properties of the HF-Dirac operator. These properties are not specific to the HF case, and the approach can be extended to completely general overlap Dirac operators obtained by substituting a general ultralocal lattice Dirac operator (involving the full Clifford algebra of γ matrices) into the overlap formula (1.8), as it was done in [20]. The full extension, which involves considerable additional work, will be carried out in a forthcoming paper [32], and it will be shown there that the axial anomaly and index of the general overlap Dirac operator D_{ov} continue to have the correct classical continuum limits in the physical (doubler-free) parameter region specified by $0 < m < \min\{M^{(j)} \neq 0\}$. It was worthwhile to consider the overlap-HF case on its own, firstly because of the current interest in using this operator in numerical simulations, and secondly because it illustrates the main ideas and techniques of the general case but without the extensive formalism and additional complications of the latter.

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